Japan. J. Math.
Vol. 19, No. 2, 1994

On deformations of locally stable holomorphic maps

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(Received May 26, 1992)
(Revised September 21, 1992)

Introduction

A *locally stable holomorphic map* is a complex analytic version of a *locally stable* or *infinitesimally stable* C^∞ *map* introduced by J.N. Mather in [4]. In C^∞ category *local stability* implies *stability* if the source manifold is compact. That is, if a C^∞ map $f : X \to Y$ between C^∞ manifolds with X compact is locally stable, then there exists an open neighborhood N of f in $C^\infty(X, Y)$ (the space of all C^∞ maps from X to Y with the so-called *Whitney* C^∞ *topology*) such that for any $g \in N$ there exist diffeomorphisms $\varphi : X \to X$ and $\psi : Y \to Y$ with $\psi \circ f \circ \varphi^{-1} = g$ (cf. [5]). Although we cannot expect this fact in complex analytic category, a locally stable holomorphic map also has a significance in complex analytic geometry as shown in [7], [8], and [9]. In this paper we shall prove two basic facts about a locally stable holomorphic map: first, that its small deformation is locally stable (Theorem 2.4), and secondly, C^∞ triviality of deformations of a locally stable holomorphic map (Theorem 3.4). Although, substantially, the former fact has already been proved in [7], we shall reproduce the proof in this paper under a little bit more general setting, supplementing a proof to an unsatisfactory point in the proof in [7]. The fact that small deformations of a locally stable holomorphic map are also locally stable is related to the existence of the *Kuranishi family for logarithmic deformations of complex analytic subspaces with locally stable parametrizations* of compact complex manifolds, as mentioned in [8].

The author expresses his heartly thanks to his colleague Prof. K. Miyajima of Kagoshima University for useful discussions with him during the preparation of this paper.

§1. Definition of locally stable holomorphic maps

Let X and Y be complex manifolds, S a finite subset of X, and q a point of Y. A multi-germ $f : (X, S) \to (Y, q)$ of a holomorphic map at S is an equivalence class

Partly supported by the Grants-in-Aid for Scientific Research (No. 03640082), The Ministry of Education, Scienceand Culture, Japan

of holomorphic maps $g : U \to Y$ with $g(S) = q$, where U are open neighborhoods of S in X. Throughout this paper we shall interchangeably use a multi-germ of f and a representative g of f. A germ of a parametrized family of multi-germs of holomorphic maps is a multi-germ $F : (X \times \mathbb{C}^r, S \times o) \to (Y \times \mathbb{C}^r, q \times o)$ of a holomorphic map such that $F(X \times t) \subset Y \times t$ for any t in some open neighborhood of the origin o in $\mathbb{C}^r$. An unfolding of a multi-germ $f : (X, S) \to (Y, q)$ of a holomorphic map is a germ of a parametrized family of multi-germs of holomorphic maps $F : (X \times \mathbb{C}^r, S \times o) \to (Y \times \mathbb{C}^r, q \times o)$ such that $F(x, o) = (f(x), o)$ for $x \in X$. We say that an unfolding $F : (X \times \mathbb{C}^r, S \times o) \to (Y \times \mathbb{C}^r, q \times o)$ of a multi-germ $f : (X, S) \to (Y, q)$ of a holomorphic map is *trivial* if there exist germs of t-levels ($t \in \mathbb{C}^r$) preserving analytic automorphisms $G : (X \times \mathbb{C}^r, S \times o) \to (X \times \mathbb{C}^r, S \times o)$ and $H : (Y \times \mathbb{C}^r, q \times o) \to (Y \times \mathbb{C}^r, q \times o)$ with $G_{|X \times o} = id_X$ and $H_{|Y \times o} = id_Y$ such that $H \circ F \circ G^{-1} = f \times id_{\mathbb{C}^r}$.

DEFINITION 1.1. A multi-germ $f : (X, S) \to (Y, q)$ of a holomorphic map is said to be *simultaneously stable* if any unfolding of f is trivial.

DEFINITION 1.2. A holomorphic map $f : X \to Y$ between complex manifolds is said to be *locally stable* if, for any point $q \in f(X)$ and any finite subset $S \subset f^{-1}(q)$, a multi-germ $f : (X, S) \to (Y, q)$ is simultaneously stable.

There is an infinitesimal criterion for a multi-germ $f : (X, S) \to (Y, q)$ of a holomorphic map to be locally stable, which is due to J.N. Mather. Now we wish to explain this fact. We denote by Θ_X (resp. Θ_Y) the sheaf of germs of holomorphic vector fields on X (resp. Y), and by $f^*\Theta_Y$ the pull-back of Θ_Y by f. We denote by $\Theta_{X,p}$ (resp. $f^*\Theta_{Y,p}$) the stalk of Θ_X (resp. $f^*\Theta_Y$) at a point p in X, and by $\Theta_{Y,q}$ the stalk of Θ_Y at a point q in Y. We denote by $tf : \Theta_{X,p} \to f^*\Theta_{Y,p}$ a $\mathcal{O}_{X,p}$-homomorphism defined by the Jacobian map $(df)_p$ and by $\omega f : \Theta_{Y,f(p)} \to f^*\Theta_{Y,p}$ a homomorphism over f^* defined by the pull-back by f, where f^* denotes the homomorphism $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ between the stalks of structure sheaves, induced by f (if $\mathbb{A}$ is an $\mathbb{R}$-module, $\mathbb{B}$ an $\mathbb{S}$-module, and $\varphi : \mathbb{R} \to \mathbb{S}$ a ring homomorphism, then we say a map $\Phi : \mathbb{A} \to \mathbb{B}$ is a homomorphism over φ if $\Phi(\alpha a + \beta b) = \varphi(\alpha)\Phi(a) + \varphi(\beta)\Phi(b)$ for all α, $\beta \in \mathbb{R}$, a, $b \in \mathbb{A}$). More generally, for any finite set $S = \{p_1, \dots, p_s\}$ of distinct points of X with $q := f(S)$, a point of Y, we define

$$
\begin{aligned}
\mathcal{O}_{X,S} &:= \mathcal{O}_{X,p_1} \times \cdots\cdots \times \mathcal{O}_{X,p_s}, \\
\Theta_{X,S} &:= \Theta_{X,p_1} \times \cdots\cdots \times \Theta_{X,p_s}, \qquad (1.1) \\
f^*\Theta_{Y,S} &:= f^*\Theta_{Y,p_1} \times \cdots\cdots \times f^*\Theta_{Y,p_s}.
\end{aligned}
$$

Then the mappings tf and ωf defined above induce a $\mathcal{O}_{X,S}$-homomorphism

$$tf : \Theta_{X,S} \longrightarrow f^*\Theta_{Y,S} \qquad (1.2)$$

and a homomorphism over f^*

$$\omega f : \Theta_{Y,q} \longrightarrow f^*\Theta_{Y,S}, \tag{1.3}$$

where f^* denotes the homomorphism $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,S}$ induced by f.

DEFINITION 1.3. A multi-germ $f : (X,S) \to (Y,q)$ of a holomorphic map is said to be *simultaneously infinitesimally stable* if

$$tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) = f^*\Theta_{Y,S} \tag{1.4}$$

holds.

We denote by $\widehat{\mathcal{O}}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{Y,q}$) the formal power series ring at $p \in X$ (resp. $q \in Y$). We define $\widehat{\Theta}_{X,p} := \widehat{\mathcal{O}}_{X,p} \otimes_{\mathcal{O}_{X,p}} \Theta_{X,p}$ and $\widehat{\Theta}_{Y,q} := \widehat{\mathcal{O}}_{Y,q} \otimes_{\mathcal{O}_{Y,q}} \Theta_{Y,q}$. The mappings $tf : \widehat{\Theta}_{X,S} \to f^*\widehat{\Theta}_{Y,S}$ and $\omega f : \widehat{\Theta}_{Y,q} \to f^*\widehat{\Theta}_{Y,S}$ are also defined in the same way as above. We define

$$\mathcal{M}_S^l := \mathfrak{m}_{p_1}^l \times \cdots \times \mathfrak{m}_{p_s}^l \quad \text{and} \quad \widehat{\mathcal{M}}_S^l := \widehat{\mathfrak{m}}_{p_1}^l \times \cdots \times \widehat{\mathfrak{m}}_{p_s}^l$$

for a natural number l, where $\mathfrak{m}_{p_i}$ and $\widehat{\mathfrak{m}}_{p_i}$ $(1 \le i \le s)$ denote the maximal ideals of $\mathcal{O}_{X,p_i}$ and $\widehat{\mathcal{O}}_{X,p_i}$, respectively.

THEOREM 1.4. (J.N. Mather, S. Tsuboi) *A multi-germ $f : (X,S) \to (Y,q)$ of a holomorphic map is simultaneously stable if and only if it satisfies one of the following mutually equivalent conditions*:

(i) $tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) = f^*\Theta_{Y,S}$

(ii) $tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) + (f^*\mathfrak{m}_q + \mathcal{M}_S^{m+1})f^*\Theta_{Y,S} = f^*\Theta_{Y,S}$ $(m := \dim Y)$

(iii) $tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) + \mathcal{M}_S^{m+1}f^*\Theta_{Y,S} = f^*\Theta_{Y,S}$

(iv) $tf(\widehat{\Theta}_{X,S}) + \omega f(\widehat{\Theta}_{Y,q}) + \widehat{\mathcal{M}}_S^{m+1}f^*\widehat{\Theta}_{Y,S} = f^*\widehat{\Theta}_{Y,S}$

(v) $tf(\widehat{\Theta}_{X,S}) + \omega f(\widehat{\Theta}_{Y,q}) + (f^*\widehat{\mathfrak{m}}_q + \widehat{\mathcal{M}}_S^{m+1})f^*\widehat{\Theta}_{Y,S} = f^*\widehat{\Theta}_{Y,S}$

(vi) $tf(\widehat{\Theta}_{X,S}) + \omega f(\widehat{\Theta}_{Y,q}) = f^*\widehat{\Theta}_{Y,S}$.

For the proof we refer to J.N. Mather [6, Theorem (1.13)] and S. Tsuboi [7, Chapter I, §2].

§2. Small deformations of locally stable holomorphic maps

DEFINITION 2.1. By a *family of holomorphic maps parametrized by a complex space*, we mean a sextuple $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of complex spaces $\mathcal{X}$, $\mathcal{Y}$, M and holomorphic maps $F : \mathcal{X} \to \mathcal{Y}$, $\pi_1 : \mathcal{X} \to M$, $\pi_2 : \mathcal{Y} \to M$ satisfying the following conditions:

(i) π_1, π_2 are surjective smooth holomorphic maps,
(ii) $\pi_1 = \pi_2 \circ F$.

For a family $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of holomorphic maps parametrized by a complex space, we call M its *parameter space*.

DEFINITION 2.2. For a given holomorphic map $f : X \to Y$ between complex manifolds, by a *family of deformations of* $f : X \to Y$ *parametrized by a complex space*, we mean a ninetuple $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M, o, \varphi, \psi)$ satisfying the following conditions:

(i) $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ is a family of holomorphic maps parametrized by a complex space M,
(ii) o is an assigned point of M,
(iii) $\varphi : X \xrightarrow{\simeq} \pi_1^{-1}(o)$ and $\psi : Y \xrightarrow{\simeq} \pi_2^{-1}(o)$ are biholomorphic maps for which $\psi^{-1} \circ F_{|\pi_1^{-1}(o)} \circ \varphi = f$ holds, where $F_{|\pi_1^{-1}(o)} : \pi_1^{-1}(o) \to \pi_2^{-1}(o)$ denotes the restrictions of F to the fibre $\pi_1^{-1}(o)$.

We introduce some notions associated to a family $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of holomorphic maps parametrized by a complex space M. We define

$$T_{\mathcal{X}/M} := \operatorname{Ker}\{d\pi_1 : T_{\mathcal{X}} \longrightarrow \pi_1^* T_M\}$$

$$(\text{resp. } T_{\mathcal{Y}/M} := \operatorname{Ker}\{d\pi_2 : T_{\mathcal{Y}} \longrightarrow \pi_2^* T_M\}),$$

where $T_{\mathcal{X}}$ (resp. $T_{\mathcal{Y}}$) and T_M denote the (holomorphic) tangent spaces of $\mathcal{X}$ (resp. $\mathcal{Y}$) and M, respectively; $d\pi_1 : T_{\mathcal{X}} \to \pi_1^* T_M$ (resp. $d\pi_2 : T_{\mathcal{Y}} \to \pi_2^* T_M$) denote the Jacobian map of the map $\pi_1 : \mathcal{X} \to M$ (resp. $\pi_2 : \mathcal{Y} \to M$). We call $T_{\mathcal{X}/M}$ (resp. $T_{\mathcal{Y}/M}$) the *(holomorphic) tangent space along fibers of the family* $\pi_1 : \mathcal{X} \to M$ (resp. $\pi_2 : \mathcal{Y} \to M$) *of complex manifolds*. By definition it is a holomorphic vector bundle over $\mathcal{X}$ (resp. over $\mathcal{Y}$). In the following, for a complex space, say Z, and a vector bundle over Z, say F, we denote by $\mathcal{O}(F)$ the sheaf of germs of holomorphic cross-sections of F. We define $\Theta_{\mathcal{X}/M} := \mathcal{O}(T_{\mathcal{X}/M})$ (resp. $\Theta_{\mathcal{Y}/M} := \mathcal{O}(T_{\mathcal{Y}/M})$). We call it the *sheaf of germs of holomorphic vector fields along fibers of the family* $\pi_1 : \mathcal{X} \to M$ (resp. $\pi_2 : \mathcal{Y} \to M$) *of complex manifolds*. We denote by $tF : \Theta_{\mathcal{X}} \to F^*\Theta_{\mathcal{Y}}$ the homomorphism of $\mathcal{O}_{\mathcal{X}}$-modules induced by the Jacobian map. $dF : T_{\mathcal{X}} \to F^*T_{\mathcal{Y}}$, where $\Theta_{\mathcal{X}} := \mathcal{O}(T_{\mathcal{X}})$ and $\Theta_{\mathcal{Y}} := \mathcal{O}(T_{\mathcal{Y}})$. The map tF induces naturally a homomorphism of $\mathcal{O}_{\mathcal{X}}$-modules from $\Theta_{\mathcal{X}/M}$ to $F^*\Theta_{\mathcal{Y}/M}$, which we denote by

$$\widehat{tF} : \Theta_{\mathcal{X}/M} \longrightarrow F^*\Theta_{\mathcal{Y}/M}. \tag{2.1}$$

We denote by $\omega F : \Theta_{\mathcal{Y}} \to F_*(F^*\Theta_{\mathcal{Y}})$ the homomorphism of $\mathcal{O}_{\mathcal{Y}}$-modules defined by the pull-back by F (cf. (1.3)). The map ωF induces naturally a homomorphism of $\mathcal{O}_{\mathcal{Y}}$-modules from $\Theta_{\mathcal{Y}/M}$ to $F_*(F^*\Theta_{\mathcal{Y}/M})$, which we denote by

$$\widehat{\omega F} : \Theta_{\mathcal{Y}/M} \longrightarrow F_*(F^*\Theta_{\mathcal{Y}/M}). \tag{2.2}$$

For a point q of $F(\mathcal{X})$, and S a finite subset of $F^{-1}(q)$, we define $\Theta_{\mathcal{X}/M,S}$ and $F^*\Theta_{\mathcal{Y}/M,S}$ as in (1.1). Then, as in (1.2) and (1.3) the mappings $\widehat{tF}$ and $\widehat{\omega F}$ induce a $\mathcal{O}_{\mathcal{X},S}$-homomorphism

$$\Theta_{\mathcal{X}/M,S} \longrightarrow F^*\Theta_{\mathcal{Y}/M,S} \tag{2.3}$$

and a homomorphism over the homomorphism $F^* : \mathcal{O}_{\mathcal{Y},q} \to \mathcal{O}_{\mathcal{X},S}$

$$\Theta_{\mathcal{Y}/M,q} \longrightarrow F^*\Theta_{\mathcal{Y}/M,S}, \tag{2.4}$$

which we denote by the same symbol as $\widehat{tF}$ and $\widehat{\omega F}$, respectively, in the following.

LEMMA 2.3. *Let $f : X \to Y$ be a locally stable holomorphic map between complex manifolds and $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M, o, \varphi, \psi)$ a family of deformations of $f : X \to Y$, parametrized by a complex space M. Let q be a point of $F(X_o)$, where $X_o := \pi_1^{-1}(o)$, and S a finite subset of $F^{-1}(q)$. Then we have*

$$\widehat{tF}(\Theta_{\mathcal{X}/M,S}) + \widehat{\omega F}(\Theta_{\mathcal{Y}/M,q}) = F^*\Theta_{\mathcal{Y}/M,S}. \tag{2.5}$$

PROOF. In order to prove the lemma, it suffices to restrict our considerations to a multi-germ of a holomorphic map $F : (\mathcal{X}, S) \to (\mathcal{Y}, q)$, which is the germ of a parametrized family of multi-germs of holomorphic maps, whose parameter space is the germ of a complex space (M, o). Hence we may assume that $\mathcal{X} = X \times M$, $\mathcal{Y} = Y \times M$; $q = q' \times o$, $S = S' \times o$, where q' is a point of $f(X)$, S' a finite subset of $f^{-1}(q')$, M a closed complex subspace of a domain D in a complex number space $\mathbb{C}^r$, and o the origin of $\mathbb{C}^r$. Let $G : (X \times \mathbb{C}^r, S) \to (Y \times \mathbb{C}^r, q)$ be an unfolding of a multi-germ $f : (X, S') \to (Y, q')$ which generates $F : (\mathcal{X}, S) \to (\mathcal{Y}, q)$, i.e., the restriction of G to $\mathcal{X}$ coincides with F. We define $\mathcal{X}' := X \times \mathbb{C}^r$ and $\mathcal{Y}' := Y \times \mathbb{C}^r$. Then we have the following commutative diagrams:

$$\begin{array}{ccc} \Theta_{\mathcal{X}'/\mathbb{C}^r,S} & \xrightarrow{\widehat{tG}} & G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S} \\ \downarrow & & \downarrow \\ \Theta_{\mathcal{X}/M,S} & \xrightarrow{\widehat{tF}} & F^*\Theta_{\mathcal{Y}/M,S} \end{array} \tag{2.6}$$

$$\begin{array}{ccc} \Theta_{\mathcal{Y}'/\mathbb{C}^r,q} & \xrightarrow{\widehat{\omega G}} & G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S} \\ \downarrow & & \downarrow \\ \Theta_{\mathcal{Y}/M,q} & \xrightarrow{\widehat{\omega F}} & F^*\Theta_{\mathcal{Y}/M,S} \end{array} \tag{2.7}$$

Since all vertical arrows in (2.6) and (2.7) are surjective, the equality

$$\widehat{tG}(\Theta_{\mathcal{X}'/\mathbb{C}^r,S}) + \widehat{\omega G}(\Theta_{\mathcal{Y}'/\mathbb{C}^r,q}) = G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S} \tag{2.8}$$

implies that of the lemma. Hence we shall prove this equality. In the following we identify $X \times o$, $Y \times o$, $S = S' \times o$, $q = q' \times o$, and $G_o := G_{|X\times o} : X \times o \to Y \times o$ with X, Y, S', q', and $f : X \to Y$, respectively. There is an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules

$$0 \longrightarrow \pi_1'^*\mathcal{I}_o G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r} \longrightarrow G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r} \xrightarrow{\alpha} f^*\Theta_Y \longrightarrow 0, \tag{2.9}$$

where π_1' denotes the projection $\mathcal{X}' := X \times \mathbb{C}^r \to \mathbb{C}^r$ and $\mathcal{I}_o$ the ideal sheaf of the origin o of $\mathbb{C}^r$ in $\mathcal{O}_{\mathbb{C}^r}$. Hence we have an isomorphism of $\mathcal{O}_{\mathcal{X}'}$-modules

$$f^*\Theta_Y \simeq G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r}/\pi_1'^*\mathcal{I}_o G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r}. \tag{2.10}$$

Similarly, we have isomorphisms

$$\Theta_X \simeq \Theta_{\mathcal{X}'/\mathbb{C}^r}/\pi_1'^*\mathcal{I}_o\Theta_{\mathcal{X}'/\mathbb{C}^r} \quad \text{and} \tag{2.11}$$

$$\Theta_Y \simeq \Theta_{\mathcal{Y}'/\mathbb{C}^r}/\pi_2'^*\mathcal{I}_o\Theta_{\mathcal{Y}'/\mathbb{C}^r}, \tag{2.12}$$

where π_2' denotes the projection $\mathcal{Y}' := Y \times \mathbb{C}^r \to \mathbb{C}^r$. It is easy to see that the homomorphism $\alpha : G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r} \to f^*\Theta_Y$ in (2.9) maps $\widehat{tG}(\Theta_{\mathcal{X}'/\mathbb{C}^r,S})$ onto $tf(\Theta_{X,S})$ and $(G^*\mathfrak{m}_q)(G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S})$ onto $(f^*\mathfrak{m}_q')(f^*\Theta_{Y,S})$. Here $\mathfrak{m}_q$ denotes the maximal ideal of $\mathcal{O}_{\mathcal{Y}',q}$ and $\mathfrak{m}_q'$ that of $\mathcal{O}_{Y,q}$; $G^*\mathfrak{m}_q$ the image of $\mathfrak{m}_q$ by the homomorphism $G^* : \mathcal{O}_{\mathcal{Y}',q} \to \mathcal{O}_{\mathcal{X}',S}$; $(G^*\mathfrak{m}_q)(G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S})$ the image of $G^*\mathfrak{m}_q \times G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S}$ by the map $\mathcal{O}_{\mathcal{X}',S} \times G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S} \to G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S}$; $(f^*\mathfrak{m}_q')(f^*\Theta_{Y,S})$ the image of $f^*\mathfrak{m}_q' \times f^*\Theta_{Y,S}$ by the map $\mathcal{O}_{X,S} \times f^*\Theta_{Y,S} \to f^*\Theta_{Y,S}$. Therefore, since $(\pi_1'^*\mathcal{I}_o G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r})_S \subset (G^*\mathfrak{m}_q)(G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S})$, by (2.10) we obtain an isomorphism of $\mathcal{O}_{\mathcal{X}',S}$-modules

$$\begin{aligned} &G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S}/\{\widehat{tG}(\Theta_{\mathcal{X}'/\mathbb{C}^r,S}) + (G^*\mathfrak{m}_q)(G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S})\} \\ &\quad \simeq f^*\Theta_{Y,S}/\{tf(\Theta_{X,S}) + (f^*\mathfrak{m}_q')(f^*\Theta_{Y,S})\}. \end{aligned} \tag{2.13}$$

Since f is locally stable, the multi-germ $f : (X,S) \to (Y,q)$ is simultaneously stable; hence $tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) = f^*\Theta_{Y,S}$ (cf. Theorem 1.4). Therefore, the natural $\mathbb{C}$ $(= \mathcal{O}_{Y,q}/\mathfrak{m}_q')$-homomorphism from $\Theta_{Y,q}/\mathfrak{m}_q'\Theta_{Y,q}$ to the latter module in (2.13), which is induced by $\omega f : \Theta_{Y,q} \to f^*\Theta_{Y,S}$, is surjective. Hence a finite $\mathbb{C}$-basis for the latter module in (2.13) is given by the projection of $\omega f(B)$ for some finite subset B of $\Theta_{Y,q}$. By the isomorphism in (2.12), B is the projection of a finite subset B' of $\Theta_{\mathcal{Y}'/\mathbb{C}^r,q}$. Then, by the isomorphism in (2.13), $\widehat{\omega G}(B')$ must be projected to a $\mathbb{C}$-basis for the former module in (2.13). Therefore, by the generalized preparation theorem (cf. [2, Theorem 3.6 in Chapter IV, Lemma 1.4 in Chapter V]), we conclude that the projection of $\widehat{\omega G}(B')$ generates the $\mathcal{O}_{\mathcal{X}',S}$-module $G^*\Theta_{\mathcal{Y}'/\mathbb{C}^r,S}/\widehat{tG}(\Theta_{\mathcal{X}'/\mathbb{C}^r,S})$ as a $\mathcal{O}_{\mathcal{Y}',q}$-module. This means that the equality in (2.8) certainly holds. Q.E.D.

THEOREM 2.4. *Let $f : X \to Y$ be a locally stable holomorphic map between complex manifolds with Y compact (X is not necessary to be compact) and $\mathcal{F} = (\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M, o, \varphi, \psi)$ a family of deformations of $f : X \to Y$ parametrized by a complex space M. We denote by $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ the holomorphic tangent bundles of $\mathcal{X}$ and $\mathcal{Y}$, respectively. We define*

$$\sum := \{x \in \mathcal{X} \mid \textit{the Jacobian map } (dF)_x : T_{\mathcal{X},x} \longrightarrow T_{\mathcal{Y},F(x)} \\ \textit{is not surjective}\},$$

which is an analytic subset of $\mathcal{X}$. We equip it with the structure of a reduced complex space. We assume that: $F_{|\Sigma} : \Sigma \to \mathcal{Y}$ is a proper map. Then there exists an open neighborhood M' of o in M such that for any $t \in M'$ the map $F_t := F_{|X_t} : X_t \to Y_t$ ($X_t := \pi_1^{-1}(t)$, $Y_t := \pi_2^{-1}(t)$) is a locally stable holomorphic map.

PROOF. In place of the family $\mathcal{F}$ we consider its reduction $\mathcal{F}_{\text{red}} := (\mathcal{X}_{\text{red}}, F_{\text{red}}, \mathcal{Y}_{\text{red}}, \pi_{1\text{red}}, \pi_{2\text{red}}, M_{\text{red}}, o, \varphi, \psi)$ and prove the theorem for $\mathcal{F}_{\text{red}}$. This is justified because of the following reasoning: since the map $F_o := F_{|X_o} : X_o \to Y_o (X_o := \pi_1^{-1}(o)$, $Y_o := \pi_2^{-1}(o))$ is equivalent to $f : X \to Y$, by assumption, it is locally stable. Hence, for any point $x \in \Sigma \cap X_o$, there exist open neighborhoods $\mathcal{U}_x$ of x in $\mathcal{X}$, $\mathcal{V}_{F(x)}$ of $F(x)$ in $\mathcal{Y}$, and biholomorphic maps $\varphi_x : \mathcal{U}_x \to U_x \times N_x$ and $\psi_x : \mathcal{V}_{F(x)} \to V_{F(x)} \times N_x$ over N_x such that the diagram

$$\tag{2.14} \begin{array}{ccc} \mathcal{U}_x & \xrightarrow{\varphi_x} & U_x \times N_x \\ \Big\downarrow {\scriptstyle F_{|\mathcal{U}_x}} & & \Big\downarrow {\scriptstyle F_{|U_x} \times id_{N_x}} \\ \mathcal{V}_{F(x)} & \xrightarrow{\psi_x} & V_{F(x)} \times N_x \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle Pr_{N_x}} & \\ & N_x & \end{array}$$

commutes, where $U_x := \mathcal{U}_x \cap X_o$, $V_{F(x)} := \mathcal{V}_{F(x)} \cap Y_o$ and $N_x := \pi_1(\mathcal{U}_x) = \pi_2(\mathcal{V}_{F(x)})$. It is obvious that

$$\tag{2.15} \varphi_x(\sum \cap \mathcal{U}_x) = \{\sum\nolimits_o \cap U_x\} \times (N_x)_{\text{red}},$$

where

$$\textstyle\sum_o := \{x \in X_o \mid \text{the Jacobian map } (dF_o) : T_{X_o,x} \longrightarrow T_{Y_o,F_o(x)} \\ \text{is not surjective.}\}$$

These arguments are also applicable for the family $\mathcal{F}_{\mathrm{red}}$. Hence, if we define

$$\textstyle\sum' := \{x \in \mathcal{X}_{\mathrm{red}} \mid \text{the Jacobian map } (dF_{\mathrm{red}})_x : T_{\mathcal{X}_{\mathrm{red}},x} \longrightarrow T_{\mathcal{Y}_{\mathrm{red}},F(x)} \\ \text{is not surjective}\},$$

we have

$$(\varphi_x)_{\mathrm{red}}\left(\textstyle\sum' \cap (\mathcal{U}_x)_{\mathrm{red}}\right) = \left\{\textstyle\sum_o \cap U_x\right\} \times (N_x)_{\mathrm{red}}.$$

Therefore Σ coincides with Σ' as a reduced complex space. Furthermore, the maps $F_t : X_t \to Y_t$ and $(F_{\mathrm{red}})_t : X_t \to Y_t$ are the same ones for any point $t \in M$. Consequently, it suffices to prove the theorem under the assumption that M is reduced. Hereafter, we assume this.

First, we shall show that there exists an open neighborhood M''' of o in M such that

$$F_{|\Sigma\cap\pi_1^{-1}(M''')} : \textstyle\sum \cap \pi_1^{-1}(M''') \longrightarrow \pi_2^{-1}(M''')$$

is a finite map. In the following we use a symbol

$$R(g)_p := \mathcal{O}_{Z,p}/g^*\mathfrak{m}_q\mathcal{O}_{Z,p}$$

for any holomorphic map $g : Z \to W$ between complex spaces and a point $p \in Z$, where we put $q := g(p) \in W$, and $\mathfrak{m}_q$ denotes the maximal ideal of $\mathcal{O}_{W,q}$. Let x be a point of $\Sigma \cap X_o$, for which we consider the diagram in (2.14). Then, by the commutativity of the diagram in (2.14) and by (2.15) we have

$$F_{|\Sigma\cap\mathcal{U}_x} = \psi_{F(x)}^{-1} \circ (F_{o|\Sigma_o\cap U_x} \times id_{(N_x)}) \circ \varphi_x$$

on $\Sigma \cap \mathcal{U}_x$; so for any point $x' \in \Sigma \cap \mathcal{U}_x$

$$R(F_{|\Sigma})_{x'} \simeq R(F_{o|\Sigma_o})_{x''} \tag{2.16}$$

where x'' denotes the U_x-component of $\varphi_x(x')$. Since the map $F_o = F_{|X_o} : X_o \to Y_o$ is locally stable, $F_{|\Sigma_o} : \Sigma_o \to Y_o$ is a finite map ([7, Corollary (4.1)]). Hence we have $\dim_{\mathbb{C}} R(F_{o|\Sigma_o})_{x''} < \infty$ ([1, Theorem (1.11)]); hence by the isomorphism in (2.16), $\dim_{\mathbb{C}} R(F_{|\Sigma})_{x'} < \infty$ for any point x' in $\Sigma \cap \mathcal{U}_x$. Since the collection $\{\mathcal{U}_x\}_{x\in\Sigma\cap X_o}$ covers $\Sigma \cap X_o$, and since $\Sigma \cap X_o$ is compact, we may extract a finite subcover indexed by $x_1, \ldots, x_k$. Since $F_{|\Sigma} : \Sigma \to \mathcal{Y}$ and $\pi_2 : \mathcal{Y} \to M$ are proper maps by assumption, so is $\pi_{1|\Sigma} : \Sigma \to M$. Hence there exists an open neighborhood M''' of o in M such that $\Sigma \cap \pi_1^{-1}(M''') \subset \bigcup_{i=1}^{k} \mathcal{U}_{x_i}$. Then the way to choose $\mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_k}$ ensures $\dim_{\mathbb{C}} R(F_{|\Sigma})_x < \infty$ for any point x in $\Sigma \cap \pi_1^{-1}(M''')$. Therefore we conclude that $F_{|\Sigma \cap \pi_1^{-1}(M''')} : \Sigma \cap \pi_1^{-1}(M''') \to \pi_2^{-1}(M''')$ is a finite map ([1, Theorem (1.11)]).

We define $\mathcal{G} := F^*\Theta_{\mathcal{Y}/M}/\widehat{tF}(\Theta_{\mathcal{X}/M})$ (cf. (2.1)). Next, we shall show that the direct image $F_*(\mathcal{G})$ of $\mathcal{G}$ by F is a coherent $\mathcal{O}_{\mathcal{Y}}$-module over $\pi_2^{-1}(M'')$ for any relatively compact open neighborhood M'' of o in M with $M'' \subset M'''$. We define $\mathcal{I} := \mathcal{I}_\Sigma$, the ideal sheaf of Σ in $\mathcal{O}_{\mathcal{X}}$. We should note that $\operatorname{Supp} \mathcal{G} \subset \Sigma$, where $\operatorname{Supp} \mathcal{G}$ denotes the support of the coherent sheaf $\mathcal{G}$. Since $\pi_{1|\Sigma} = \pi_2 \circ F_{|\Sigma} : \Sigma \to M$ is a proper map, $\pi_1^{-1}(M'') \cap \Sigma$ is relatively compact. Hence, by *Rückert's Nullstellensatz* ([3, Chapter 2, §2]), there exists a natural number N such that $\mathcal{I}^N \mathcal{G} = 0$ on $\pi_1^{-1}(M'')$. We consider the following exact sequences of $\mathcal{O}_{\mathcal{X}}$-modules over $\pi_1^{-1}(M'')$:

$$
(2.17)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}\mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}_{|\Sigma} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{I}^2\mathcal{G} & \longrightarrow & \mathcal{I}\mathcal{G} & \longrightarrow & \mathcal{I}\mathcal{G}_{|\Sigma} & \longrightarrow & 0 \\
 & & \cdots & \cdots & \cdots & & & & \\
 & & \cdots & \cdots & \cdots & & & & \\
0 & \longrightarrow & \mathcal{I}^N\mathcal{G} & \longrightarrow & \mathcal{I}^{N-1}\mathcal{G} & \xrightarrow{\simeq} & \mathcal{I}^{N-1}\mathcal{G}_{|\Sigma} & \longrightarrow & 0 \\
 & & \| & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

We claim the homomorphism of $\mathcal{O}_{\mathcal{Y}}$-modules $F_*(\mathcal{I}^l\mathcal{G}) \to (F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G}_{|\Sigma})$ is surjective over $\pi_2^{-1}(M'')$ for any l with $0 \le l \le N-1$. Indeed, let q be any point in $F(\Sigma) \cap \pi_2^{-1}(M'')$, and $\{p_1, \dots, p_s\}$ the set of all distinct points $F^{-1}(q) \cap \Sigma$. Then we have an isomorphism of stalks

$$\{(F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G}_{|\Sigma})\}_q \simeq \prod_{i=1}^{s} (\mathcal{I}^l\mathcal{G}_{|\Sigma})_{p_i}.$$

We choose Stein open neighborhoods $U_1, \dots, U_s$ of $p_1, \dots, p_s$ in $\mathcal{X}$, respectively, which are mutually disjoint. Let

$$a_q = (b_{1,p_1}, \dots, b_{s,p_s}) \in \{(F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G}_{|\Sigma})\}_q \simeq \prod_{i=1}^{s} (\mathcal{I}^l\mathcal{G}_{|\Sigma})_{p_i}$$

be given. Take a cross-section $a \in \Gamma(V \cap F(\Sigma), (F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G}_{|\Sigma}))$ which represents a_q at q, where V is a Stein open neighborhood of q in $\mathcal{Y}$. If we take V sufficiently small, we may assume that $F^{-1}(V) \cap \Sigma \subset \bigcup_{i=1}^{s} U_i$. Let $b_i \in \Gamma(F^{-1}(V) \cap U_i \cap \Sigma, \mathcal{I}^l\mathcal{G}_{|\Sigma})$ $(1 \le i \le s)$ be a cross-section which represents b_{i,p_i} at p_i. Since $F^{-1}(V) \cap U_i$ is a Stein open subset ([3, p.33 ~ p.34]), there exists a cross-section $\widetilde{b}_i \in \Gamma(F^{-1}(V) \cap U_i, \mathcal{I}^l\mathcal{G})$ such that $\widetilde{b}_{i|F^{-1}(V)\cap U_i\cap\Sigma} = b_i$. Then, since $\operatorname{Supp}\mathcal{I}^l\mathcal{G} \subset \Sigma$, there exists a cross-section $\widetilde{b} \in \Gamma(F^{-1}(V), \mathcal{I}^l\mathcal{G})$ such that $\widetilde{b}$ coincides with $\widetilde{b}_i$ over $F^{-1}(V)\cap U_i$ for every i. We may consider $\widetilde{b}$ to be an element of $\Gamma(V, F_*(\mathcal{I}^l\mathcal{G}))$. It is clear $\widetilde{b}_q = a_q$, and so the sheaf homomorphism $F_*(\mathcal{I}^l\mathcal{G}) \to (F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G}_{|\Sigma})$ is certainly surjective as claimed. Therefore, taking direct images of the exact sequences of sheaves over $\pi_1^{-1}(M'')$ in (2.17), we get the following exact sequences of $\mathcal{O}_\mathcal{Y}$-modules over $\pi_2^{-1}(M'')$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_*(\mathcal{I}\mathcal{G}) & \longrightarrow & F_*(\mathcal{G}) & \longrightarrow & (F_{|\Sigma})_*(\mathcal{G}_{|\Sigma}) & \longrightarrow & 0 \\
 & \cdots & \cdots & \cdots & \cdots & & & & \\
 & \cdots & \cdots & \cdots & \cdots & & & & \\
0 & \longrightarrow & F_*(\mathcal{I}^{N-1}\mathcal{G}) & \longrightarrow & F_*(\mathcal{I}^{N-2}\mathcal{G}) & \longrightarrow & (F_{|\Sigma})_*(\mathcal{I}^{N-2}\mathcal{G}_{|\Sigma}) & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & F_*(\mathcal{I}^{N-1}\mathcal{G}) & \xrightarrow{\simeq} & (F_{|\Sigma})_*(\mathcal{I}^{N-1}\mathcal{G}_{|\Sigma}) & \longrightarrow & 0.
\end{array}
\tag{2.18}
$$

$(F_{|\Sigma})_*(\mathcal{I}^l\mathcal{G})$ is a coherent $\mathcal{O}_\mathcal{Y}$-module for every l $(0 \le l \le N-1)$, because $F_{|\Sigma} : \Sigma \to \mathcal{Y}$ is a proper map by assumption. Hence, by the last exact sequence in (2.18) we conclude $F_*(\mathcal{I}^{N-1}\mathcal{G})$ is coherent over $\pi_2^{-1}(M'')$. Then, by the one preceding the last exact sequence in (2.18), so is $F_*(\mathcal{I}^{N-2}\mathcal{G})$. Continuing this argument successively, we conclude $F_*(\mathcal{G})$ is coherent over $\pi_2^{-1}(M'')$ as asserted.

Finally, we shall show that there exists an open neighborhood M' of o in M'' such that $F_t : X_t \to Y_t$ is locally stable for any $t \in M'$. We define a homomorphism of $\mathcal{O}_\mathcal{Y}$-modules $\widehat{\omega F} : \Theta_{\mathcal{Y}/M} \to F_*(\mathcal{G})$ to be the composite of $\widehat{\omega F} : \Theta_{\mathcal{Y}/M} \to F_*(F^*\Theta_{\mathcal{Y}/M})$ (cf. (2.2)) and the map $F_*(F^*\Theta_{\mathcal{Y}/M}) \to F_*(\mathcal{G})$. We define $\mathcal{H} := F_*(\mathcal{G})/\widehat{\omega F}(\Theta_{\mathcal{Y}/M})$. $\mathcal{H}$ is a coherent $\mathcal{O}_\mathcal{Y}$-module over $\pi_2^{-1}(M'')$. Hence $\operatorname{Supp}(\mathcal{H}_{|\pi_2^{-1}(M'')})$ $(\subset F(\Sigma)\cap\pi_2^{-1}(M''))$ is an analytic subset of $\pi_2^{-1}(M'')$. We claim that $\mathcal{H}_q$ (the stalk of $\mathcal{H}$ at q) $= 0$ for any point $q \in Y_o := \pi_2^{-1}(o)$. Indeed, if we define $\Sigma_q := F^{-1}(q)\cap\Sigma = F_o^{-1}(q)\cap\Sigma_o$ for a point $q \in Y_o$, then

$$
\mathcal{H}_q \simeq F^*\Theta_{\mathcal{Y}/M,\Sigma_q}/\{\widehat{tF}(\Theta_{\mathcal{X}/M,\Sigma_q}) + \widehat{\omega F}(\Theta_{\mathcal{Y}/M,q})\}. \tag{2.19}
$$

Since $F_o := F_{|X_o} : X_o \to Y_o$ is locally stable, by Lemma 2.3 the right hand side in (2.19) vanishes. Therefore we conclude that $\mathcal{H}_q = 0$ for any point $q \in Y_o$. This means $Y_o \cap \operatorname{Supp}\mathcal{H} = \emptyset$. Since $\pi_2 : \mathcal{Y} \to M$ is a proper map, $\pi_2(\operatorname{Supp}(\mathcal{H}_{|\pi_2^{-1}(M'')}))$ is an analytic subset of M''; hence, since $Y_o\cap\operatorname{Supp}\mathcal{H} = \emptyset$, $o \notin \pi_2(\operatorname{Supp}\mathcal{H})$. Therefore there exists an open neighborhood M' of o in M'' such that $M'\cap\pi_2(\operatorname{Supp}\mathcal{H}) = \emptyset$, so $\pi_2^{-1}(M')\cap\operatorname{Supp}\mathcal{H} = \emptyset$. This means $\mathcal{H}_q = 0$ for any point $q \in \pi_2^{-1}(M')$; equivalently

$$
\widehat{tF}(\Theta_{\mathcal{X}/M,\Sigma_q}) + \widehat{\omega F}(\Theta_{\mathcal{Y}/M,q}) = F^*\Theta_{\mathcal{Y}/M,\Sigma_q} \tag{2.20}
$$

holds for any point $q \in \pi_2^{-1}(M')$, where $\Sigma_q := F^{-1}(q)\cap\Sigma$ (cf. (2.19)). By (2.20) we have

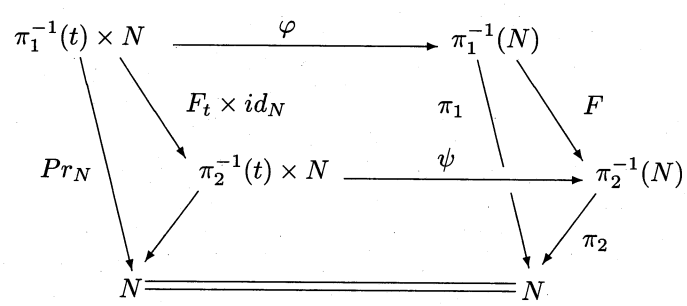
$$tF_t(\Theta_{X_t,\Sigma_q}) + \omega F_t(\Theta_{Y_t,q}) = F_t^*\Theta_{Y_t,\Sigma_q} \tag{2.21}$$

for any $t \in M'$ and any $q \in Y_t$. Furthermore, the equality which is obtained by replacing Σ_q in (2.21) by any finite subset S of $F^{-1}(q)$ also holds, because the map $tF_t : \Theta_{X_t,p} \to F_t^*\Theta_{Y_t,p}$ is surjective for any point p in $X_t \backslash \Sigma$. Therefore we conclude that F_t is locally stable for any $t \in M'$ (cf. Theorem 1.4). Q.E.D.

§3. C^∞ triviality of deformations of locally stable holomorphic maps

We define a C^∞ *family of* C^∞ *maps parametrized by a* C^∞ *manifold* by replacing complex analytic objects by corresponding C^∞ ones in Definition 2.1. For a C^∞ family $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of C^∞ maps parametrized by a C^∞ manifold, we define the C^∞ *tangent bundle along fibers* $T_{\mathcal{X}/M}$ (resp. $T_{\mathcal{Y}/M}$) *of the family* $\pi_1 : \mathcal{X} \to M$ (resp. $\pi_2 : \mathcal{Y} \to M$) *of* C^∞ *manifolds* in the same way as in complex analytic case. In the following, for a C^∞ manifold, say Z, we denote by T_Z the C^∞ tangent bundle of Z, and for C^∞ vector bundle on Z, say F, we denote by $\mathcal{A}(F)$ the sheaf of germs of C^∞ cross-sections of F. Then, as in complex analytic case, for a C^∞ family $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of C^∞ maps parametrized by a C^∞ manifold, the sheaf homomorphisms $tF : \mathcal{A}(T_{\mathcal{X}}) \to F^*\mathcal{A}(T_{\mathcal{Y}})$, $\omega F : \mathcal{A}(T_{\mathcal{Y}}) \to F_*(F^*\mathcal{A}(T_{\mathcal{Y}}))$, $\widehat{tF} : \mathcal{A}(T_{\mathcal{X}/M}) \to F^*\mathcal{A}(T_{\mathcal{Y}/M})$, and $\widehat{\omega F} : \mathcal{A}(T_{\mathcal{Y}/M}) \to F_*(F^*\mathcal{A}(T_{\mathcal{Y}/M}))$ are defined.

Definition 3.1. We say a C^∞ family $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ of C^∞ maps parametrized by a C^∞ manifold is C^∞ *trivial at* $t \in M$ if there exist an open neighborhood N of t in M and diffeomorphisms $\varphi : \pi_1^{-1}(t) \times N \to \pi_1^{-1}(N)$, $\psi : \pi_2^{-1}(t) \times N \to \pi_2^{-1}(N)$ such that the diagram



commutes, where $F_t := F_{|\pi_1^{-1}(t)} : \pi_1^{-1}(t) \to \pi_2^{-1}(t)$ denotes the restriction of F to $\pi_1^{-1}(t)$.

We quote a proposition from [2], which gives a sufficient condition for a C^∞ family of C^∞ maps parametrized by a C^∞ manifold to be C^∞ trivial.

PROPOSITION 3.2. *Let $\mathcal{F} = (\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, D)$ be a C^∞ family of C^∞ maps between compact C^∞ manifolds parametrized by a domain D of a real number space $\mathbb{R}^m$. We denote by $(t_1, \dots, t_m)$ a coordinate system on $\mathbb{R}^m$. Suppose that for every i $(1 \leq i \leq m)$ there exist C^∞ vector fields along fibers*

$$\zeta^i \in \Gamma(\mathcal{X}, \mathcal{A}(T_{\mathcal{X}/D})) \quad \textit{and} \quad \eta^i \in \Gamma(\mathcal{Y}, \mathcal{A}(T_{\mathcal{Y}/D}))$$

such that

$$\widehat{tF}(\zeta^i) + \widehat{\omega F}(\eta^i) = tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right),$$

where $\widehat{tF}$, $\widehat{\omega F}$, tF, ωF denote the homomorphisms between global cross-sections induced by the corresponding sheaf homomorphisms. Then the family $\mathcal{F}$ is C^∞ trivial at any point $t \in D$.

For the proof we refer to Theorem 3.3 (Thom-Levine) in [2, Chapter V].

There is an analogous proposition which gives a sufficient condition for a (complex analytic) family of holomorphic maps to be C^∞ trivial. In order to explain this fact we introduce some symbols. Let $(\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, M)$ be a (complex analytic) family of holomorphic maps parametrized by a complex manifold. We denote by $\mathcal{X}_\mathbb{R}$ (resp. $\mathcal{Y}_\mathbb{R}$, resp. $M_\mathbb{R}$) the underlying C^∞ manifold of $\mathcal{X}$ (resp. $\mathcal{Y}$, resp. M), and by $F_\mathbb{R} : \mathcal{X}_\mathbb{R} \to \mathcal{Y}_\mathbb{R}$ (resp. $\pi_{1\mathbb{R}} : \mathcal{X}_\mathbb{R} \to M_\mathbb{R}$, resp. $\pi_{2\mathbb{R}} : \mathcal{Y}_\mathbb{R} \to M_\mathbb{R}$) the underlying C^∞ map of $F : \mathcal{X} \to \mathcal{Y}$ (resp. $\pi_1 : \mathcal{X} \to M$, resp. $\pi_2 : \mathcal{Y} \to M$). We denote by $T_{\mathcal{X}/M}$ and $\overline{T_{\mathcal{X}/M}}$ the holomorphic tangent bundle along fibers of the complex analytic family $\pi_1 : \mathcal{X} \to M$ of complex manifolds and its conjugate holomorphic bundle, respectively. We denote by $T_{\mathcal{X}_\mathbb{R}/M_\mathbb{R}}$ the C^∞ tangent bundle along fibers of the C^∞ family $\pi_{1\mathbb{R}} : \mathcal{X}_\mathbb{R} \to M_\mathbb{R}$ of C^∞ manifolds. The relation among these bundles is given by $T_{\mathcal{X}_\mathbb{R}/M_\mathbb{R}} \otimes_\mathbb{R} \mathbb{C} = T_{\mathcal{X}/M} \oplus \overline{T_{\mathcal{X}/M}}$. We also denote by $T_{\mathcal{Y}/M}$, $\overline{T_{\mathcal{Y}/M}}$ and $T_{\mathcal{Y}_\mathbb{R}/M_\mathbb{R}}$ the corresponding vector bundles of the complex analytic family $\pi_2 : \mathcal{Y} \to M$ of complex manifolds.

PROPOSITION 3.3. *Let $\mathcal{F} = (\mathcal{X}, F, \mathcal{Y}, \pi_1, \pi_2, D)$ be a (complex analytic) family of holomorphic maps between compact complex manifolds, parametrized by a domain D of a complex number space $\mathbb{C}^m$. We denote by $(t_1, \dots, t_m)$ a coordinate system on $\mathbb{C}^m$. Suppose that for every i $(1 \leq i \leq m)$ there exist C^∞ global cross-sections*

$$\zeta^i \in \Gamma(\mathcal{X}, \mathcal{A}(T_{\mathcal{X}/D})) \quad \textit{and} \quad \eta^i \in \Gamma(\mathcal{Y}, \mathcal{A}(T_{\mathcal{Y}/D}))$$

such that

$$\widehat{tF}(\zeta^i) + \widehat{\omega F}(\eta^i) = tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right),$$

where $\widehat{tF}$, $\widehat{\omega F}$, tF, ωF denote the homomorphisms between global cross-sections induced by the corresponding sheaf homomorphisms. Then $\mathcal{F}$'s underlying C^∞ family $\mathcal{F}_\mathbb{R} := (\mathcal{X}_\mathbb{R}, F_\mathbb{R}, \mathcal{Y}_\mathbb{R}, \pi_{1\mathbb{R}}, \pi_{2\mathbb{R}}, D_\mathbb{R})$ of C^∞ maps is C^∞ trivial at any point $t \in D_\mathbb{R}$.

PROOF. We shall show that if the condition of the proposition is satisfied by the family $\mathcal{F}$, then the condition of Proposition 3.2 is satisfied by the underlying C^∞ family $\mathcal{F}_\mathbb{R}$ of $\mathcal{F}$. We denote the extensions of $tF_\mathbb{R}$, $\omega F_\mathbb{R}$, $\widehat{tF}_\mathbb{R}$, and $\widehat{\omega F}_\mathbb{R}$ over the ground field $\mathbb{C}$ by $tF_\mathbb{C}$, $\omega F_\mathbb{C}$, $\widehat{tF}_\mathbb{C}$, and $\widehat{\omega F}_\mathbb{C}$, respectively. That is, we define

$$tF_\mathbb{C} : \mathcal{A}(T_{\mathcal{X}_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}) \longrightarrow F_\mathbb{R}^* \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}),$$

$$\omega F_\mathbb{C} : \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}) \longrightarrow F_{\mathbb{R}*}(F_\mathbb{R}^* \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C})),$$

$$\begin{aligned}\widehat{tF}_\mathbb{C} : \mathcal{A}(T_{\mathcal{X}_\mathbb{R}/D_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}) &= \mathcal{A}(T_{\mathcal{X}/D} \oplus \overline{T_{\mathcal{X}/D}}) \\ &\longrightarrow F_\mathbb{R}^* \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}/D_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}) = \mathcal{A}(F^* T_{\mathcal{Y}/D} \oplus \overline{F^* T_{\mathcal{Y}/D}}),\end{aligned}$$

and

$$\begin{aligned}\widehat{\omega F}_\mathbb{C} : \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}/D_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}) &= \mathcal{A}(T_{\mathcal{Y}/D} \oplus \overline{T_{\mathcal{Y}/D}}) \\ &\longrightarrow F_{\mathbb{R}*}(F_\mathbb{R}^* \mathcal{A}(T_{\mathcal{Y}_\mathbb{R}/D_\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C})) = F_{\mathbb{R}*}(\mathcal{A}(F^* T_{\mathcal{Y}/D} \oplus \overline{F^* T_{\mathcal{Y}/D}})).\end{aligned}$$

Let $t_i = u_i + \sqrt{-1} v_i$ $(1 \le i \le m)$ be the expression of t_i in real coordinate functions u_i, v_i. Then, by the condition we have

$$\begin{aligned}
&\widehat{tF}_\mathbb{R}(\zeta^i + \overline{\zeta^i}) + \widehat{\omega F}_\mathbb{R}(\eta^i + \overline{\eta^i}) \\
&= \widehat{tF}_\mathbb{C}(\zeta^i) + \widehat{tF}_\mathbb{C}(\overline{\zeta^i}) + \widehat{\omega F}_\mathbb{C}(\eta^i) + \widehat{\omega F}_\mathbb{C}(\overline{\eta^i}) \\
&= \widehat{tF}(\zeta^i) + \widehat{\omega F}(\eta^i) + \overline{\widehat{tF}(\zeta^i)} + \overline{\widehat{\omega F}(\eta^i)} \\
&= tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right) + \overline{tF\left(\frac{\partial}{\partial t_i}\right)} - \overline{\omega F\left(\frac{\partial}{\partial t_i}\right)} \\
&= tF_\mathbb{C}\left(\frac{\partial}{\partial t_i}\right) - \omega F_\mathbb{C}\left(\frac{\partial}{\partial t_i}\right) + tF_\mathbb{C}\left(\frac{\partial}{\partial \overline{t_i}}\right) - \omega F_\mathbb{C}\left(\frac{\partial}{\partial \overline{t_i}}\right) \\
&= tF_\mathbb{R}\left(\frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}\right) - \omega F_\mathbb{R}\left(\frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}\right) \\
&= tF_\mathbb{R}\left(\frac{\partial}{\partial u_i}\right) - \omega F_\mathbb{R}\left(\frac{\partial}{\partial u_i}\right).
\end{aligned} \tag{3.1}$$

Similarly, we have

$$
\begin{aligned}
&\widehat{tF_{\mathbb{R}}}\left(\sqrt{-1}(\zeta^i-\overline{\zeta^i})\right)+\widehat{\omega F_{\mathbb{R}}}\left(\sqrt{-1}(\eta^i-\overline{\eta^i})\right)\\
&=tF_{\mathbb{R}}\left(\sqrt{-1}\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial \overline{t_i}}\right)\right)+\omega F_{\mathbb{R}}\left(\sqrt{-1}\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial \overline{t_i}}\right)\right) \qquad (3.2)\\
&=tF_{\mathbb{R}}\left(\frac{\partial}{\partial v_i}\right)-\omega F_{\mathbb{R}}\left(\frac{\partial}{\partial v_i}\right).
\end{aligned}
$$

Since $\zeta^i+\overline{\zeta^i}$, $\sqrt{-1}(\zeta^i-\overline{\zeta^i})\in\Gamma(\mathcal{X}_{\mathbb{R}},\mathcal{A}(T_{\mathcal{X}_{\mathbb{R}}/D_{\mathbb{R}}}))$, and $\eta^i+\overline{\eta^i}$, $\sqrt{-1}(\eta^i-\overline{\eta^i})\in\Gamma(\mathcal{Y}_{\mathbb{R}},\mathcal{A}(T_{\mathcal{Y}_{\mathbb{R}}/D_{\mathbb{R}}}))$, (3.1) and (3.2) show that the condition of Proposition 3.2 is certainly satisfied by the family $\mathcal{F}_{\mathbb{R}}$. This completes the proof of the proposition.

We are in a position to prove a theorem concerning C^∞ triviality of deformations of a locally stable holomorphic map.

THEOREM 3.4. *Let $\mathcal{F}=(\mathcal{X},F,\mathcal{Y},\pi_1,\pi_2,D,o,\varphi,\psi)$ be a (complex analytic) family of deformations of a locally stable holomorphic map $f:X\to Y$ between compact complex manifolds, parametrized by a domain D of a complex number space $\mathbb{C}^m$. Then $\mathcal{F}$'s underlying C^∞ family $\mathcal{F}_{\mathbb{R}}$ of C^∞ maps is C^∞ trivial at $o\in D$.*

PROOF. By Theorem 2.4 there exists a relatively compact open neighborhood N of o in D such that $F_t:=F_{|X_t}:X_t\to Y_t$ ($X_t:=\pi_1^{-1}(t)$, $Y_t:=\pi_2^{-1}(t)$) is locally stable for any $t\in\overline{N}$. We shall show that if we take such an open neighborhood N of o in D, then for every i ($1\le i\le m$) there exist C^∞ global cross-sections

$$\zeta^i\in\Gamma(\pi_1^{-1}(N),\mathcal{A}(T_{\mathcal{X}/M}))\quad\text{and}\quad \eta^i\in\Gamma(\pi_2^{-1}(N),\mathcal{A}(T_{\mathcal{Y}/M}))$$

such that

$$\widehat{tF}(\zeta^i)+\widehat{\omega F}(\eta^i)=tF\left(\frac{\partial}{\partial t_i}\right)-\omega F\left(\frac{\partial}{\partial t_i}\right)\quad\text{on }\ \pi_1^{-1}(N), \tag{3.3}$$

where $(t_1,\dots,t_m)$ denotes a coordinate system on $\mathbb{C}^m$. Then the theorem follows from Proposition 3.3. The proof of the above assertion will proceed in three steps.

STEP I. As in Theorem 2.4 we define

$$\sum:=\{x\in\mathcal{X}\mid \text{the Jacobian map } (dF)_x:T_{\mathcal{X},x}\longrightarrow T_{\mathcal{Y},F(x)}\ \text{is not surjective}\},$$

which is an analytic subset of $\mathcal{X}$. We equip it with the structure of a reduced complex space. We shall show that for an open neighborhood N of o in D, taken as above, there exist a finite number of open subsets $U_1,\dots,U_k$ of $\mathcal{X}$, $W_1,\dots,W_k$,

$W_1', \ldots, W_k'$, $V_1, \ldots, V_k$ of $\mathcal{Y}$, and local holomorphic vector fields along fibers $\zeta_\lambda^i \in \Gamma(U_\lambda, \mathcal{O}(T_{\mathcal{X}/D}))$, $\eta_\lambda^i \in \Gamma(V_\lambda, \mathcal{O}(T_{\mathcal{Y}/D}))$ $(1 \le i \le m,\ 1 \le \lambda \le k)$ satisfying the following conditions:

$$(3.4)\quad \begin{aligned}
&\text{(i)} && F(\Sigma \cap \pi_1^{-1}(\overline{N})) \subset \textstyle\bigcup_{\lambda=1}^k W_\lambda \\
&\text{(ii)} && \overline{W}_\lambda \subset W_\lambda',\ \overline{W}_\lambda' \subset V_\lambda \qquad \text{for } \lambda = 1, \ldots, k \\
&\text{(iii)} && F^{-1}(\overline{W}_\lambda') \cap \Sigma \subset U_\lambda \qquad \text{for } \lambda = 1, \ldots, k \\
&\text{(iv)} && F(U_\lambda) \subset V_\lambda \text{ and } \widehat{tF}(\zeta_\lambda^i) + \widehat{\omega F}(\eta_\lambda^i) = tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right) \text{ on } U_\lambda \\
& && \qquad \text{for } \lambda = 1, \ldots, k,\ i = 1, \ldots, m.
\end{aligned}$$

Since $\Sigma \cap \pi_1^{-1}(\overline{N})$ is compact and the map $F : \mathcal{X} \to \mathcal{Y}$ is a proper map, the set $F(\Sigma \cap \pi_1^{-1}(\overline{N}))$ is compact. Hence it suffices to show that for any point $y_o \in F(\Sigma \cap \pi_1^{-1}(\overline{N}))$ there exist open subsets U of $\mathcal{X}$, W, W', V of $\mathcal{Y}$ with $y_o \in W$, and local holomorphic vector fields along fibers $\zeta \in \Gamma(U, \mathcal{O}(T_{\mathcal{X}/D}))$, $\eta \in \Gamma(V, \mathcal{O}(T_{\mathcal{Y}/D}))$ satisfying the conditions (ii), (iii), (iv) in (3.4). Since $F_t := F_{|X_t} : X_t \to Y_t$ is locally stable for any $t \in \overline{N}$, $F_{t|\Sigma_t} : \Sigma_t \to Y_t$ $(\Sigma_t := \Sigma \cap \pi_1^{-1}(t))$ is a finite map (cf. [7, Corollary 4.1]); so $\Sigma(y_o) := \Sigma \cap F^{-1}(y_o)$ $(\subset \Sigma_{\pi_2(y_o)})$ is a finite set. Therefore by Lemma 2.3 there exist open neighborhoods U of $\Sigma(y_o)$ in $\mathcal{X}$, V of y_o in $\mathcal{Y}$ with $F(U) \subset V$, and local holomorphic vector fields along fibers $\zeta^i \in \Gamma(U, \mathcal{O}(T_{\mathcal{X}/D}))$, $\eta^i \in \Gamma(V, \mathcal{O}(T_{\mathcal{Y}/D}))$ $(1 \le i \le m)$ such that the condition (iv) in (3.4) holds on U.

Next, we shall show that there exists an open neighborhood W' of y_o in $\mathcal{Y}$ such that $F^{-1}(\overline{W}') \cap \Sigma \subset U$. Indeed, if it does not exist, we can take a sequence $\{x_n\}$ of points in $\Sigma \backslash U$ with $\lim_{n\to\infty} F(x_n) = y_o$. Let K be a compact neighborhood of y_o in $\mathcal{Y}$. Then there exists a natural number N such that if $n \ge N$, then $F(x_n) \in K$; hence $x_n \in F^{-1}(K) \cap \Sigma$ for $n \ge N$. Since K is compact and F is a proper map, $F^{-1}(K)$ is compact; so is $F^{-1}(K) \cap \Sigma$. Hence we can choose a subsequence $\{x_m'\}$ of $\{x_n\}$ which converges to a point, say x_o, of $\mathcal{X}$. Since $\{x_m'\}$ is a sequence of points in $\Sigma \backslash U$, and since $\Sigma \backslash U$ is closed, we have $x_o \in \Sigma \backslash U$. On the other hand, since $F(x_o) = \lim_{m\to\infty} F(x_m') = \lim_{n\to\infty} F(x_n) = y_o$, we have $x_o \in \Sigma(y_o)$. But this is a contradiction, because $(\Sigma \backslash U) \cap \Sigma(y_o) = \emptyset$. Therefore we conclude that there exists an open neighborhood W' of y_o in $\mathcal{Y}$ such that $F^{-1}(\overline{W}') \cap \Sigma \subset U$. We take W' so small that it satisfies the condition (ii) in (3.4) with respect to V; and take an open neighborhood $W$of y_o in $\mathcal{Y}$ with $\overline{W} \subset W'$, then we are done.

Step II. We claim that there exists an open neighborhood $\mathcal{U}_0$ of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$ such that $F^{-1}(\overline{W}_\lambda') \cap \mathcal{U}_0 \subset U_\lambda$ for every λ $(1 \le \lambda \le k)$. In order to prove this it suffices to show the existence of such an open neighborhood $\mathcal{U}_0$ for a fixed λ. If there does not exist such an open neighborhood $\mathcal{U}_0$, then for any open neighborhood $\mathcal{U}$ of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$ we can take a point $x \in F^{-1}(\overline{W}_\lambda') \cap \mathcal{U} \cap (\mathcal{X} \backslash U_\lambda)$. Let L be a compact neighborhood of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$, and $\{\mathcal{X}_\alpha\}_{\alpha=1,2,\ldots}$ an infinite system of open neighborhoods of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in L with $\bigcap_{\alpha=1}^{\infty} \mathcal{X}_\alpha = \Sigma \cap \pi_1^{-1}(\overline{N})$. We take a point x_α from each $F^{-1}(\overline{W}'_\lambda) \cap \mathcal{X}_\alpha \cap (\mathcal{X} \backslash U_\lambda)$ $(\alpha = 1, 2, \dots)$ and form a sequence $\{x_\alpha\}_{\alpha=1,2,\dots}$ of points in L. Then, since L is compact, we can take a subsequence $\{x'_\beta\}_{\beta=1,2,\dots}$ of $\{x_\alpha\}_{\alpha=1,2,\dots}$ which converges to a point, say x_o, of L. We denote by $\mathcal{X}'_\beta$ an open neighborhood $\mathcal{X}_\alpha$ for which $x'_\beta = x_\alpha$. Then, since $\bigcap_{\beta=1}^{\infty} \mathcal{X}'_\beta = \Sigma \cap \pi_1^{-1}(\overline{N})$, we have $x_o \in \Sigma \cap \pi_1^{-1}(\overline{N})$. On the other hand, since the sequence $\{x'_\beta\}_{\beta=1,2,\dots}$ is contained in both of the two closed subsets $F^{-1}(\overline{W}'_\lambda)$ and $\mathcal{X} \backslash U_\lambda$, we have $x_o \in F^{-1}(\overline{W}'_\lambda) \cap (\mathcal{X} \backslash U_\lambda)$. But this is a contradiction, because $F^{-1}(\overline{W}'_\lambda) \cap \Sigma \subset U_\lambda$ by the condition (iii) in (3.4). Therefore we conclude that the claim certainly holds.

STEP III. Let $U_1, \dots, U_k$ be open subsets of $\mathcal{X}$, $W_1, \dots, W_k$, $W'_1, \dots, W'_k$, $V_1, \dots, V_k$ those of $\mathcal{Y}$, and $\zeta^i_\lambda \in \Gamma(U_\lambda, \mathcal{O}(T_{\mathcal{X}/D}))$, $\eta^i_\lambda \in \Gamma(V_\lambda, \mathcal{O}(T_{\mathcal{Y}/D}))$ $(1 \le i \le m$, $1 \le \lambda \le k)$ local holomorphic vector fields along fibers, which satisfy the conditions (i) $\sim$ (iv) in (3.4). Let $\mathcal{U}_0$ be an open neighborhood of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$ such that $F^{-1}(\overline{W}'_\lambda) \cap \mathcal{U}_0 \subset U_\lambda$ for every λ $(1 \le \lambda \le k)$. We put $W'_0 := \mathcal{Y} \backslash \mathcal{W}$ and $V_0 := \mathcal{Y}$, where we define $\mathcal{W} := \bigcup_{\lambda=1}^{k} W_\lambda$. We take a C^∞ partition of unity $\{\rho_\lambda\}_{\lambda=0,1,\dots,k}$ on $\mathcal{Y}$, subordinate to the covering $\{W'_\lambda\}_{\lambda=0,1,\dots,k}$, i.e., ρ_λ's are C^∞ functions on $\mathcal{Y}$ satisfying the following conditions:

(i) $0 \le \rho_\lambda \le 1$ for $\lambda = 0, 1, \dots, k$

(ii) $\operatorname{Supp} \rho_\lambda \subset W'_\lambda$ for $\lambda = 0, 1, \dots, k$

(iii) $\sum_{\lambda=0}^{k} \rho_\lambda \equiv 1$ on $\mathcal{Y}$.

Note that $\sum_{\lambda=1}^{k} \rho_\lambda \equiv 1$ on $\mathcal{W} := \bigcup_{\lambda=1}^{k} W_\lambda$, since $\operatorname{Supp} \rho_0 \cap \mathcal{W} = \emptyset$. We take a relatively compact open neighborhood $\mathcal{U}'_0$ of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{U}_0$ with $\overline{\mathcal{U}'_0} \subset \mathcal{U}_0$ and a C^∞ function ρ on $\mathcal{X}$ satisfying: (i) $\operatorname{Supp} \rho \subset \mathcal{U}_0$,and (ii) $\rho \equiv 1$ on $\mathcal{U}'_0$. We define

$$\zeta^i := \sum_{\lambda=1}^{k} \rho F^*(\rho_\lambda) \zeta^i_\lambda \qquad (i = 1, \dots, m) \tag{3.5}$$

$$\eta^i := \sum_{\lambda=0}^{k} \rho_\lambda \eta^i_\lambda \qquad (i = 1, \dots, m) \tag{3.6}$$

where we understand η^i_0 to be the zero form on W'_0. We should note that, since $\operatorname{Supp} \rho F^*(\rho_\lambda) \subset F^{-1}(\overline{W}'_\lambda) \cap \mathcal{U}_0 \subset U_\lambda$ and $\operatorname{Supp} \rho_\lambda \subset W'_\lambda \subset V_\lambda$, the expressions on the right hand sides in (3.5) and (3.6) make sense. ζ^i and η^i are global C^∞ vector fields on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Furthermore, we have

$$\widehat{tF}(\zeta^i) + \widehat{\omega F}(\eta^i)$$

$$(3.7) \qquad = \sum_{\lambda=1}^{k}(\rho F^*(\rho_\lambda)\widehat{tF}(\zeta^i_\lambda)) + \sum_{\lambda=1}^{k}(F^*(\rho_\lambda)\widehat{\omega F}(\eta^i_\lambda))$$

$$= \sum_{\lambda=1}^{k}\{F^*(\rho_\lambda)(\rho\widehat{tF}(\zeta^i_\lambda) + \widehat{\omega F}(\eta^i_\lambda))\} \quad \text{on } \mathcal{U}_0 \qquad (i = 1, \dots, m).$$

Here we should note that, since $\operatorname{Supp} F^*(\rho_\lambda) \subset F^{-1}(\overline{W}'_\lambda)$ and $\mathcal{U}_0 \cap F^{-1}(\overline{W}'_\lambda) \subset U_\lambda$ for every λ $(1 \leq \lambda \leq k)$, the last expression in (3.7) makes sense on $\mathcal{U}_0$. Since $\rho \equiv 1$ on $\mathcal{U}'_0$ and $\sum_{\lambda=1}^{k} F^*(\rho_\lambda) \equiv 1$ on $F^{-1}(\mathcal{W})$, by (3.7) and the condition (iv) in (3.4) we have

$$(3.8) \qquad \widehat{tF}(\zeta^i) + \widehat{\omega F}(\eta^i) = tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right) \quad \text{on } \mathcal{U}'_0 \cap F^{-1}(\mathcal{W}) \quad (i = 1, \dots, m).$$

Since $\Sigma \cap \pi^{-1}(\overline{N}) \subset \mathcal{U}'_0$, and since, $F(\Sigma \cap \pi_1^{-1}(\overline{N})) \subset \mathcal{W}$ by the condition (i) in (3.4), we have

$$(3.9) \qquad \Sigma \cap \pi_1^{-1}(\overline{N}) \subset \mathcal{U}'_0 \cap F^{-1}(\mathcal{W}).$$

In case $\dim X < \dim Y$: We have $\Sigma = \mathcal{X}$. Hence, by (3.9) $\pi_1^{-1}(\overline{N}) \subset \mathcal{U}'_0 \cap F^{-1}(\mathcal{W})$. Therefore, by (3.8) we conclude that the assertion of the theorem certainly holds.

In case $\dim X \geq \dim Y$: We define

$$\theta_i^F := -\widehat{tF}\left(\zeta^i - \frac{\partial}{\partial t_i}\right) - \widehat{\omega F}\left(\eta^i + \frac{\partial}{\partial t_i}\right) \qquad (i = 1, \dots, m)$$

and

$$\mathcal{U}_1 := \mathcal{U}'_0 \cap F^{-1}(\mathcal{W}).$$

Then $\mathcal{U}_1$ is an open neighborhood of $\Sigma \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$, and by (3.8) $\theta_i^F \equiv 0$ on $\mathcal{U}_1$. Since $F_{|\mathcal{X}\setminus\Sigma} : \mathcal{X}\setminus\Sigma \to \mathcal{Y}$ is a submersion, for any point $x \in (\mathcal{X}\setminus\mathcal{U}_1) \cap \pi_1^{-1}(\overline{N})$ there exist an open neighborhood U_x of x in $\mathcal{X}\setminus\Sigma$ and local holomorphic vector fields along fibers $\xi_x^i \in \Gamma(U_x, \mathcal{O}(T_{\mathcal{X}/D}))$ with $\widehat{tF}(\xi_x^i) = \theta_i^F$ on U_x for every i $(1 \leq i \leq m)$. Since $(\mathcal{X}\setminus\mathcal{U}_1) \cap \pi_1^{-1}(\overline{N})$ is compact, we may extract a finite subcover $\{U_{x_\mu}\}_{\mu=1,\dots,l}$ of a covering $\{U_x\}_{x \in (\mathcal{X}\setminus\mathcal{U}_1)\cap\pi_1^{-1}(\overline{N})}$ of $(\mathcal{X}\setminus\mathcal{U}_1) \cap \pi_1^{-1}(\overline{N})$ in $\mathcal{X}$. In the following we write U_μ and ξ_μ for U_{x_μ} and ξ_{x_μ}, respectively. We put $U_0 := \mathcal{U}_1$ and define $\mathcal{X}' := \bigcup_{\mu=0}^{l} U_\mu$.

We take a C^∞ partition of unity $\{\tau_\mu\}_{\mu=0,1,\dots,l}$ on $\mathcal{X}'$, subordinate to the covering $\{U_\mu\}_{\mu=0,1,\dots,l}$, and define

$$\xi^i := \sum_{\mu=0}^{l} \tau_\mu \xi^i_\mu \qquad (i = 1, \dots, m),$$

where we understand ξ^i_0 to be the zero form on U_0. Then $\xi^i \in \Gamma(\mathcal{X}', \mathcal{A}(T_{\mathcal{X}/D}))$; and since $\widehat{tF}(\xi^i_\mu) = \theta^F_i$ on U_μ for every i and μ ($1 \leq i \leq m$, $0 \leq \mu \leq l$), we have

$$\widehat{tF}(\xi^i) = \sum_{\mu=0}^{l} \tau_\mu \widehat{tF}(\xi^i_\mu) = \sum_{\mu=0}^{l} \tau_\mu \theta^F_i = \theta^F_i \quad \text{on } \mathcal{X}' \qquad (i = 1, \dots, m). \tag{3.10}$$

Hence, by the definition of θ^F_i we have

$$\widehat{tF}(\xi^i + \zeta^i) + \widehat{\omega F}(\eta^i) = tF\left(\frac{\partial}{\partial t_i}\right) - \omega F\left(\frac{\partial}{\partial t_i}\right) \quad \text{on } \mathcal{X}' \qquad (i = 1, \dots, m).$$

Since $\xi^i + \zeta^i \in \Gamma(\mathcal{X}', \mathcal{A}(T_{\mathcal{X}/D}))$, $\eta^i \in \Gamma(\mathcal{Y}, \mathcal{A}(T_{\mathcal{Y}/D}))$ and $\pi_1^{-1}(N) \subset \mathcal{X}'$, this completes the proof of the theorem.

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